Infinite symmetry in the fractional quantum Hall effect

1994 J. Phys. A: Math. Gen. 27 3999

(http://iopscience.iop.org/0305-4470/27/11/045)

J. Phys. A: Math. Gen. 27 (1994) 3999–4010. Printed in the UK

Infinite symmetry in the fractional quantum Hall effect

Michael Flohr†‡ and Raimund Varnhagen†‡
Physikalisches Institut der Universität Bonn, Nussallee 12, D-53115 Bonn, Germany

Received 4 January 1994

Abstract. We have generalized recent results on the integer quantum Hall effect, constructing explicitly a $\mathcal{W}_{1+\infty}$ for the fractional quantum Hall effect such that the negative modes annihilate the Laughlin wavefunctions. This generalization has a nice interpretation in Jain's composite-fermion theory. Furthermore, for these models, we have calculated the wavefunctions of the edge excitations, viewing them as area-preserving deformations of an incompressible quantum droplet and have shown that the $\mathcal{W}_{1+\infty}$ is the underlying symmetry of the edge excitations in the fractional quantum Hall effect. Finally, we have applied this method to more general wavefunctions.

1. Introduction

The experimental discoveries of the integer quantum Hall effect (IQHE) [1] and of the fractional quantum Hall effect (FQHE) [2] are some of the most interesting physical phenomena in solid-state physics in recent years. The conductance of a two-dimensional electron gas in a high-magnetic field at low temperature exhibits quantized plateau values of the form $\sigma_{xy} = (e^2/h)\nu$ where the filling factor ν is an integer or fractional number. In many respects, both the integer and the fractional effect share very similar underlying physical characteristics and concepts, for instance the two-dimensionality of the system, the quantization of the Hall conductance with simultaneous vanishing of the longitudinal resistance and the interplay between disorder and the magnetic field giving rise to the existence of extended states. In other respects, they encompass entirely different physical principles and ideas. In particular, while the IQHE is thought of essentially as a non-interacting electron phenomenon [3], the FQHE is believed to arise from a condensation of the two-dimensional electrons into a new incompressible state of matter as a result of interelectron interaction [4], see also [5, 6].

An important step was taken by Laughlin [4] who wrote down the wavefunctions for the fundamental fractions $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ which played a special role in a hierarchial scheme in which a daughter state was obtained at each step from a condensation of quasiparticles of the parent state into a correlated low-energy state [7, 8]. Extensive calculations have proven these wavefunctions to be extremely close to the numerical exact solutions [5].

In recent years, Jain [9] developed the composite-fermion theory which could describe IQHE and FQHE by a common principle, attaching to each electron an even number of magnetic flux quanta which gives an easy explanation of the experimentally observed fractional fillings as well as a new derivation of Laughlin's wavefunctions starting from the well understood IQHE. New experiments are in good agreement with this theory [10–12].

† Supported by the Deutsche Forschungsgemeinschaft.
‡ email: unp055@ibm.rhrz.uni-bonn.de, unp034@ibm.rhrz.uni-bonn.de

0305-4470/94/113999+12$19.50

The incompressibility of these quantum fluids is explained by a finite energy gap above the ground state. Recently, for the IQHE ($\nu = 1$), it was shown that incompressibility also results in an infinite symmetry which describes the area-preserving non-singular deformations of the quantum droplet and commutes with the Hamiltonian [13]. The quantization of this symmetry is well known in physics as the non-singular part of a $\mathcal{W}_{1+\infty}$ and arises, e.g., in string theories or two-dimensional gravity [14–16]. These deformations are directly related to edge excitations which should live on the one-dimensional boundaries and were studied by a number of authors [17–22]. The dynamics of these edge states is mainly based on the relation of Chern–Simons gauge theories and conformal-field theory [23].

In this paper, we give a generalization of this infinite symmetry to the FQHE ($\nu = 1/(2p+1)$) showing that the Laughlin wavefunctions are annihilated by the negative non-singular generators of the $\mathcal{W}_{1+\infty}$. The very interesting point is that, when constructing the $\mathcal{W}_{1+\infty}$ for the FQHE, interelectron interaction effects enter which automatically cancel out for the IQHE and agree with the result of [13]. Furthermore, we can show that these interactions can be interpreted as arising from an even number of magnetic-flux quanta which are attached to the electrons as in the composite-fermion picture. It turns out that the interaction is hidden in a non-trivial measure which is the N-point function of N localized flux-quanta vortices and should be described by an Abelian Chern–Simons theory.

Viewing the QHE states as a droplet of an incompressible quantum fluid, the gapless edge excitations can be interpreted as coming from surface waves or area-preserving deformations of the droplet. We have calculated the wavefunctions for edge excitations with $\nu = 1/(2p+1)$ using the fact that they are generated by the positive modes of the $\mathcal{W}_{1+\infty}$. Here our result agrees with former ones by Stone [24] for $\nu = 1$ and Wen [18] for the FQHE.

Finally, we apply the previous method to more general wavefunctions describing multi-layer systems or systems of interacting Landau levels for every fractional filling and show that the $\mathcal{W}_{1+\infty}$ is indeed the fundamental symmetry of the edge excitations.

The paper is organized as follows: first we give an introduction to the basics of the QHE. Next, we show how to generalize the construction of the $\mathcal{W}_{1+\infty}$ from the IQHE to the FQHE and interpret this generalization by Jain's composite-fermion theory. Then we calculate the wavefunctions of the edge excitations using the $\mathcal{W}_{1+\infty}$. Finally, we consider the case of more general wavefunctions.

2. Preliminaries

Let us start by reviewing some elementary facts about a two-dimensional electron in a uniform transverse magnetic field B. The Schrödinger equation for such an electron is given by

$$H\psi = \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 \psi = E\psi \tag{2.1}$$

where the momentum $p = -\mathrm{i}\hbar\nabla$ and the gauge potential A exist in the plane. This problem can be solved exactly. Let us choose the symmetric gauge $A = (B/2)(-y, x)$ and introduce complex variables: $z = x + \mathrm{i}y$, $\bar{z} = x - \mathrm{i}y$ and $\partial = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$. Defining all lengths in units of the magnetic length

$$l = \left(\frac{2\hbar c}{eB}\right)^{1/2} \tag{2.2}$$

and the energies in units of the Landau-level spacing

$$\omega_c = \frac{eB}{mc} \tag{2.3}$$

the Hamiltonian can be re-expressed as

$$H = 2\hbar\omega_c l^2 \left(-\partial\bar{\partial} + \frac{1}{2l^2}(\bar{z}\bar{\partial} - z\partial) + \frac{1}{4l^4} z\bar{z} \right). \tag{2.4}$$

Letting $\hbar = m = l = 1$, the Hamiltonian and the angular momentum J can be written in terms of a pair of independent harmonic oscillators

$$H = a^\dagger a + a a^\dagger \tag{2.5}$$

$$J = b^\dagger b - a^\dagger a \tag{2.6}$$

where these operators are

$$a = \frac{z}{2} + \bar{\partial} \qquad a^\dagger = \frac{\bar{z}}{2} - \partial \tag{2.7}$$

$$b = \frac{\bar{z}}{2} + \partial \qquad b^\dagger = \frac{z}{2} - \bar{\partial} \tag{2.8}$$

and satisfy the commutation relations

$$[a, a^\dagger] = 1 \qquad [b, b^\dagger] = 1 \tag{2.9}$$

with all other commutators vanishing. The vacuum is determined by the condition $a\psi_{0,0} = b\psi_{0,0} = 0$ and given as

$$\psi_{0,0} = \frac{1}{\sqrt{\pi}} \exp(-\tfrac{1}{2}|z|^2). \tag{2.10}$$

In terms of the operators $a^\dagger$ and $b^\dagger$, the solutions can finally be written as

$$\psi_{n,l} = \frac{(b^\dagger)^l (a^\dagger)^n}{\sqrt{l!n!}} \psi_{0,0} \tag{2.11}$$

with energy $E_n = 2n + 1$ which determines the Landau level. These energy states are infinitely degenerate due to the rotational invariance around the z-axis. It is useful to note that in the lowest Landau level the polynomial part of the wavefunction is holomorphic and in the second Landau level involves, at most, one power of $\bar{z}$. In general, the highest power of $\bar{z}$ in the nth Landau level is $n - 1$.

In a finite sample of area A, one can show that the degeneracy of each Landau level is determined by the number of the magnetic-flux quanta

$$N_A = \frac{\Phi_{\text{mag}}}{\Phi_0} \tag{2.12}$$

where $\Phi_{\text{mag}} = BA$ is the magnetic flux through the area A and $\Phi_0 = (h/e)$ is a single flux quantum.

Let us now consider the case of N such electrons. If there is no interaction between them, the many-particle problem splits into N copies of the single-particle problem. Therefore, we get N operators, identical to the single-particle operators a, b, but now labelled by an index i referring to the coordinate of the ith electron: a_i, b_i. Since the magnetic field B controls the number of states and thus the density of electrons per state, its action can be considered as an external pressure. Actually, the electron density per state is the correct quantum measure of the electron density, i.e. the filling fraction ν

$$\nu = \frac{N}{N_A}. \tag{2.13}$$

The IQHE is well understood by a gauge argument of Laughlin. Later, it was shown that the conductivity could be interpreted as the Chern character of a $U(1)$-fibre bundle over a torus [25–27] or as an element in the cyclic cohomology of a C^* algebra [28].

For the FQHE, with filling fraction $\nu = 1/(2p+1)$, Laughlin [4] found, by numerical experiments, the ground states given by the following wavefunctions:

$$\psi_p = \prod_{i<j}(z_i - z_j)^{2p+1} \exp\left(-\tfrac{1}{2}\sum_i |z_i|^2\right) \tag{2.14}$$

where p should be an integer to respect the Pauli principle. In the composite-fermion theory, this wavefunction was reinterpreted by Jain as a wavefunction not of bare single electrons but of electrons bound to an even (here $2p$) number of vortices or flux quanta. Starting with the wavefunction ϕ_n of the IQHE with filling fraction $\nu = n$, one attaches $2p$ flux quanta to each electron which is given by multiplying ϕ_n by D^{2p}

$$\psi_\nu = D^{2p}\phi_n \qquad \text{with} \qquad D = \prod_{i<j}(z_i - z_j). \tag{2.15}$$

Using mean-field arguments, this leads to an electron state in which $n^{-1} \pm 2p$ flux quanta are available to each electron. Thus, this composite-fermion state has a filling fraction [9]

$$\nu = \frac{n}{2pn \pm 1}. \tag{2.16}$$

Thus, the Laughlin wavefunctions are given for $n = 1$. When calculating some expectation values via path integrals, only closed paths contribute to the partition function because it is the trace of $\exp(-\beta H)$. Closed paths are given by exchanging electrons or by moving them around each other. The phase associated with each path has two contributions. One is the statistical phase due to the Fermi statistics of the electrons and the other one is the Aharanov–Bohm phase due to the flux enclosed in the loop. However, adding an even number of flux quanta to a fermion again gives a fermion and, also, the Aharanov–Bohm phase factor is the same because a flux quantum produces a phase factor of unity. Thus, adding an even number of flux quanta to each electron does not change the expectation values. This argument was formulated by Lopez and Fradkin [29].

3. $\mathcal{W}_{1+\infty}$ for $\nu = 1/m$

The wavefunctions of the last section should describe the condensation of electrons to new states of matter, i.e. to incompressible quantum superfluids. Normally, the incompressibility is explained by a finite energy gap above the ground state. Recently, Cappelli *et al* [13] have given another explanation of this incompressibility for the $\nu = 1$ case; they have found a $\mathcal{W}_{1+\infty}$ symmetry which is the algebra of the area-preserving non-singular diffeomorphisms commuting with the Hamiltonian of the system, defining an incompressible state now to be a highest-weight vector of the $\mathcal{W}_{1+\infty}$ [13]. They constructed the generators of the $\mathcal{W}_{1+\infty}$ in the following way:

$$\mathcal{L}_{m,n} = \sum_{i=1}^{N} (b_i^\dagger)^{m+1} (b_i)^{n+1} \qquad \text{for } n, m \geqslant -1. \tag{3.1}$$

These generators commute with the Hamiltonian of the system and fulfil the following commutation relations:

$$[\mathcal{L}_{n,m}, \mathcal{L}_{k,l}] = \sum_{s=0}^{\mathrm{Min}(m,k)} \frac{(m+1)!(k+1)!}{(m-s)!(k-s)!(s+1)!} \mathcal{L}_{n+k-s,m+l-s} - (m \leftrightarrow l, n \leftrightarrow k). \tag{3.2}$$

Then they have shown that

$$\mathcal{L}_{m,n}\psi_0 = 0 \qquad \text{for} \qquad n > m \geqslant -1 \tag{3.3}$$

which means that ψ_0 is a highest-weight vector of the algebra of area-preserving non-singular diffeomorphisms.

The aim of our paper is to generalize this result to the FQHE. We attempt this by changing the definition of the b_i, introducing an interaction term in the following way: ($b_i^\dagger$ remains unchanged)

$$b_i = \partial_i + \frac{\bar{z}_i}{2} - 2p \sum_{i \neq j} \frac{1}{z_i - z_j}. \tag{3.4}$$

For $p = 0$, one recovers the original definition for the b_i and $b_i^\dagger$ as before, so we are not changing our notation. The commutators of the b_i and $b_i^\dagger$ change in the following way

$$[b_i, b_i^\dagger] = 1 + 2p\pi \sum_{i \neq j} \delta(z_i - z_j) \tag{3.5}$$

$$[b_i, b_j^\dagger] = -2p\pi\delta(z_i - z_j) \qquad \text{for } i \neq j\,. \tag{3.6}$$

Defining the $\mathcal{L}_{mn}$ as above but with the new b_i, these new $\mathcal{L}_{mn}$ fulfil the commutation relations of the same $\mathcal{W}_{1+\infty}$ up to terms involving delta functions. In the case of fermions, which have to respect the Pauli principle, the delta functions do not contribute since the wavefunction has to approach zero for $z_i \longrightarrow z_j, i \neq j$. For the first Landau level, where the wavefunctions are holomorphic up to the exponential term, one can rewrite the operators b_i and $b_i^\dagger$ such that they act only on the holomorphic part:

$$b_i = \partial_i - 2p \sum_{i \neq j} \frac{1}{z_i - z_j} \tag{3.7}$$

$$b_i^\dagger = z_i\,. \tag{3.8}$$

Note that $b_i^\dagger$ acts just by multiplication. Thus, in the case of the first Landau level, no delta functions will occur. In the standard notation of $\mathcal{W}_{1+\infty}$, we set

$$W_n^{(s)} \sim \mathcal{L}_{n+s-2,s-2} \qquad s \geqslant 1 \qquad n \geqslant -s+1 \tag{3.9}$$

where $W_n^{(s)}$ is the nth-Fourier mode of a spin s field. After some calculations, which can be found in the appendix, one obtains the action of the modes $W_n^{(s)}$ on the Laughlin wavefunction ψ_p

$$W_n^{(s)}\psi_p = (s-1)! \sum_{1\leqslant j_0<j_1<\ldots<j_{s-1}\leqslant N} \frac{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_{j_0} & z_{j_1} & \cdots & z_{j_{s-2}} & z_{j_{s-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{j_0}^{s-2} & z_{j_1}^{s-2} & \cdots & z_{j_{s-2}}^{s-2} & z_{j_{s-1}}^{s-2} \\ z_{j_0}^{s-1+n} & z_{j_1}^{s-1+n} & \cdots & z_{j_{s-2}}^{s-1+n} & z_{j_{s-1}}^{s-1+n} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_{j_0} & z_{j_1} & \cdots & z_{j_{s-2}} & z_{j_{s-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{j_0}^{s-2} & z_{j_1}^{s-2} & \cdots & z_{j_{s-2}}^{s-2} & z_{j_{s-1}}^{s-2} \\ z_{j_0}^{s-1} & z_{j_1}^{s-1} & \cdots & z_{j_{s-2}}^{s-1} & z_{j_{s-1}}^{s-1} \end{vmatrix}} \psi_p \tag{3.10}$$

from which it immediately follows that, acting on ψ_p, the negative modes vanish

$$W_n^{(s)}\psi_p = 0 \qquad \text{for } -s < n \leqslant -1\,. \tag{3.11}$$

Moreover, the states ψ_p are eigenstates for the zero modes

$$W_0^{(s)}\psi_p = (s-1)!\binom{N}{s}\psi_p. \tag{3.12}$$

Let us emphasize this result. We have shown that the Laughlin wavefunctions are highest-weight states of the quantized algebra of non-singular area-preserving diffeomorphisms which means that these states describe incompressible quantum fluids and that all surface waves on the droplet move in the same direction. The singular deformations cannot be included in the algebra in this way since they would change the topology of the droplet. Now, we have a common formulation for $\nu = 1$ and $\nu = 1/(2p+1)$ QHE, where the IQHE is automatically described by a single-electron theory but the FQHE needs interelectron interaction in the first Landau level.

At this point the reader may worry that b_i and $b_i^\dagger$ are not Hermitian conjugate. However, we can take an inner product of the form

$$\langle \Psi_1 \mid \Psi_2 \rangle = \int \Psi_1^\dagger \mu \Psi_2 \tag{3.13}$$

where μ is given as

$$\mu(z_1, \bar{z}_1, \ldots, z_N, \bar{z}_N) = \prod_{i<j} |z_i - z_j|^{-4p}. \tag{3.14}$$

Using this inner product, b_i and $b_i^\dagger$ become Hermitian conjugate to each other. One will see that this measure is very important in the following, especially for the interpretation of the new interaction term in the b_is. Namely, by introducing the non-trivial measure, the Hamiltonian would be non-Hermitian. Thus we have to change the definition of the $a_i^\dagger$ in the following way: (a_i also remains unchanged)

$$a_i^\dagger = -\partial_i + \frac{\bar{z}_i}{2} + 2p \sum_{i \neq j} \frac{1}{z_i - z_j}. \tag{3.15}$$

The commutation relations are now given as

$$[a_i, a_i^\dagger] = 1 - 2p\pi \sum_{i \neq j} \delta(z_i - z_j) \tag{3.16}$$

$$[a_i, a_j^\dagger] = 2p\pi\delta(z_i - z_j) \qquad \text{for } i \neq j. \tag{3.17}$$

The Hamiltonian is defined as before

$$H = \sum_{i=1}^{N} (a_i^\dagger a_i + a_i a_i^\dagger) \tag{3.18}$$

and commutes with the $\mathcal{W}_{1+\infty}$ without occurrence of any delta functions. The Landau-level structure is not destroyed and the Laughlin wavefunction for $\nu = 1/(2p+1)$ is an eigenfunction in the lowest Landau level.

The configuration space for distinguishable particles is given by

$$\mathrm{C}_N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N;\ z_i \neq z_j \text{ for } i \neq j\}. \tag{3.19}$$

The $(a_i, a_i^\dagger)$ can be considered as covariant derivatives on a $U(1) \otimes \ldots \otimes U(1)$ bundle over C_N as in the paper by Verlinde on the non-Abelian Aharanov–Bohm effect [30]. Thus, the curvature is given by (3.14) which describes a constant magnetic field plus $2p$ flux quanta added to each electron. This is exactly the FQHE interpretation by Jain in the composite-fermion theory mentioned in the previous section. These flux quanta can be described in an Abelian Chern–Simons theory by localized Wilson loops. Considering the N-point function of these flux quanta localized at the positions z_i of the electrons, one sees that it is proportional to the measure μ (3.14) remembering that these Wilson loop operators can be expressed by vertex operators [31]. This explains the former observation on the relation between vertex-operator correlators and the Laughlin wavefunction [24, 32–34].

This picture is in good agreement with the argument of Lopez and Fradkin [29], stated previously, that adding an even number of flux quanta to each electron leaves all expectation values invariant. Calculating the expectation values of the Laughlin wavefunction, one also has to introduce the measure μ (3.14)

$$\int \psi_p^\dagger \mu \psi_p \,\mathrm{d}z^N. \tag{3.20}$$

It is easy to see that this expression is independent of p, thus, adding flux quanta does not change the expectation value.

Thus, in our formulation of the FQHE, we consider a Hamiltonian without explicit interelectron interaction, as in the IQHE, but describing the interaction with the help of a non-trivial measure coming from the N-point correlation function of the flux quanta in an Abelian Chern–Simons theory.

At this point, the reader may think that our picture of the FQHE is only a complicated view of the IQHE. In fact, Lopez and Fradkin [29] have explicitly shown that adding $2p$ flux quanta changes the effective magnetic field and thus, not all observables are unchanged by our transformation of b_i, $b_i^\dagger$, a_i and $a_i^\dagger$. The change of the effective magnetic field is not the only physical effect of this transformation. Verlinde [30] has already shown that a Hamiltonian similar to ours without a magnetic field acquires a non-trivial S-matrix when a substitution of the kind

$$\partial_i \to \partial_i + \alpha \sum_{i \neq j} \frac{1}{z_i - z_j} \qquad \bar{\partial}_i \to \bar{\partial}_i$$

is applied. Therefore, our transformation is not only a mathematical reformulation of the IQHE; it also describes new physical effects.

4. Edge excitations

Halperin [17] was the first to point out that the IQHE states contain gapless edge excitations which are responsible for non-trivial transport properties. Using gauge arguments, one can easily show that FQHE states also support gapless edge excitations. Wen [18] has shown that these states span a representation of a Kac–Moody current algebra and Stone [24] has described them using Schur functions or homogenous symmetric polynomials. In this section we derive these results with the help of the $\mathcal{W}_{1+\infty}$.

Viewing the QHE states as a droplet of an incompressible quantum fluid, we consider the edge excitations as area-preserving deformations of the droplet which are described by the $\mathcal{W}_{1+\infty}$. Thus, the highest-weight representation on the QHE wavefunction should give the spectrum of these edge states

$$W_{n_1}^{(s_1)} W_{n_2}^{(s_2)} \ldots W_{n_k}^{(s_k)} \psi_p \qquad s_i \geqslant s_{i+1};\, n_i \geqslant n_{i+1} \qquad \text{if } s_i = s_{i+1}. \tag{4.1}$$

In fact, equation (3.10) shows that applying one mode of a $\mathcal{W}_{1+\infty}$ generator to ψ_p yields ψ_p multiplied by a symmetric function since the fraction of the determinants is equal to the Schur function $\mathcal{S}^{\{0,0,\ldots,0,n\}}$. In fact, every Schur function can be written as a fraction of a certain determinant and the Vandermonde determinant. If we use the notation

$$\mathcal{D}^{\{m_1,m_2,\ldots,m_n\}} = \begin{vmatrix} z_1^{m_1} & z_2^{m_1} & \cdots & z_n^{m_1} \\ z_1^{1+m_2} & z_2^{1+m_2} & \cdots & z_n^{1+m_2} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1+m_n} & z_2^{n-1+m_n} & \cdots & z_n^{n-1+m_n} \end{vmatrix} \tag{4.2}$$

then the Schur functions can be expressed as [24]

$$\mathcal{S}^{\{m_1,m_2,\ldots,m_n\}} = \frac{\mathcal{D}^{\{m_1,m_2,\ldots,m_n\}}}{\mathcal{D}^{\{0,0,\ldots,0\}}}. \tag{4.3}$$

By induction, it follows that monomials, as in (4.1), also yield polynomial symmetric functions multiplied by ψ_p. The reason is that any such state must be totally antisymmetric due to the Pauli principle. But since ψ_p is always reproduced, the only way to get a totally antisymmetric polynomial function is to multiply ψ_p by a totally symmetric one.

Moreover, the current $j \equiv W^{(1)}$ already yields a complete set of symmetric functions, namely the products of power sums $s_k = \sum_{i=1}^{N} z_i^k$

$$j_{n_1} j_{n_2} \dots j_{n_k} \psi_p = s_{n_1} s_{n_2} \dots s_{n_k} \psi_p. \tag{4.4}$$

This is the basis of all symmetric functions provided $n_i \geqslant n_{i+1}$. To see this, one has to note that the action of

$$W_k^{(1)} = \sum_{i=1}^{N} (b_i^\dagger)^k \tag{4.5}$$

on $f(z_1, \dots, z_N) \exp(-\frac{1}{2}\sum_{i=1}^{N} |z_i|^2)$, where $f(z_1, \dots, z_N)$ is any holomorphic function on $\mathbb{C}^N$, is just given by the multiplication with s_k which is a holomorphic function for $k \geqslant 0$.

Thus, the $\mathcal{W}_{1+\infty}$ algebra yields all possible edge excitations which respect the Pauli principle. The resulting spectrum is given by the set of all symmetric polynomial functions with the partition function being nothing but

$$Z(q = \mathrm{e}^{2\pi i \tau}) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{4.6}$$

where $p(n)$ denotes the number of partitions of n in positive integers. Thus, the positive modes of the current $j \equiv W^{(1)}$ alone generate all edge excitations which means that these excitations can be interpreted as surface waves moving in the same direction and moving with the same velocity. Therefore, the spectrum is equivalent to that of the $U(1)$-Kac–Moody algebra at level one. In this way, the results of Wen [18] and Stone [24] reappear in a unified way.

These considerations show that the conformal-field theory, which corresponds to the Chern–Simons theory describing the attachment of flux quanta to the electrons and which is defined on the boundary of the system (the Laughlin droplet), must be generated by a $U(1)$-Kac–Moody current. Thus, it follows that the conformal theory must have an effective central charge $c_{\mathrm{eff}} = 1$. This agrees with the fact that the non-trivial measure introduced in the third section, where it arose from the non-flat Knizhnik–Zamolodchikov connection describing the effect of the attached flux quanta, is given by a correlation function of a $c_{\mathrm{eff}} = 1$ conformal-field theory.

5. Generalizations

There exist many other examples of trial wavefunctions, not only for filling fraction $\nu = 1/(2p+1)$. Most of these wavefunctions have the following structure [18, 24, 35, 36]:

$$\psi_K = \prod_{I<J} \prod_{i \leqslant j} (z_i^I - z_j^J)^{K_{I,J}} \prod_{I} \prod_{i<j} (z_i^I - z_j^I)^{K_{I,I}} \exp\left(-\frac{1}{2} \left(\sum_{i,I} |z_i^I|^2 \right) \right) \tag{5.1}$$

where K is a symmetric integer-valued $m \times m$ matrix with odd integers on the main diagonal. Then, the filling fraction is given by

$$\nu = \sum_{I,J} (K^{-1})_{I,J}. \tag{5.2}$$

Thus, one can get different wavefunctions for the same filling fraction ν. The physical picture behind this ansatz is to couple different independent Hall fluids (i.e. sets of eventually interacting Landau levels or different layers). Viewing the filling fraction ν as being proportional to the Hall conductivity σ, one sees that the total Hall conductivity is determined by the Hall conductivities of the several Hall fluids (or Landau levels) according to the Kirchhoff rules for coupling them in parallel or in series. bye Now, it is easy to see that the $\mathcal{W}_{1+\infty}$ can be constructed in the same way as before defining b_i^I and $b_i^{I\dagger}$ as

$$b_i^I = \partial_i^I + \frac{1}{2}\bar{z}_i^I + \sum_{J\neq I} A_{I,J} \sum_{i\leqslant j} \frac{1}{z_i^I - z_j^J} + A_{I,I} \sum_{i<j} \frac{1}{z_i^I - z_j^I} \tag{5.3}$$

$$b_i^{I\dagger} = -\bar{\partial}_i^I + \tfrac{1}{2} z_i^I \tag{5.4}$$

and

$$\mathcal{L}_{m,n} = \sum_{I,i} (b_i^{I\dagger})^{m+1} (b_i^I)^{n+1} . \tag{5.5}$$

The heighest weight condition can be fulfilled, if

$$K = \mathbb{1} - A \tag{5.6}$$

where $\mathbb{1}$ is the $m \times m$ identity matrix. For example, the $\nu = m/(2pm+1)$ FQHE can be obtained if K is given by the following $m \times m$ matrix [21, 35]

$$K = \begin{pmatrix} 2p+1 & 2p & \dots & 2p \\ 2p & 2p+1 & & 2p \\ \vdots & & \ddots & \vdots \\ 2p & \dots & 2p & 2p+1 \end{pmatrix} . \tag{5.7}$$

Thus, the matrix A is nothing but

$$A = -2p \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix} \tag{5.8}$$

which indeed can be considered as the addition of $2p$ magnetic-flux quanta to each particle as stated previously.

The edge excitations are generated by the action of the $\mathcal{W}_{1+\infty}$ in a completely analogous manner. But now, if $m > 1$, the current $j \equiv W^{(1)}$ contained in the $\mathcal{W}_{1+\infty}$ is no longer sufficient to generate all the edge excitations. The partition function (4.6) has to be replaced by its mth power, i.e. the edge excitations are generated by m currents [18]. In the same way it is possible to reproduce the hierarchy picture of Haldane and Halperin [21, 35].

6. Conclusion

In this paper we have shown that the $\mathcal{W}_{1+\infty}$ is the underlying symmetry in the IQHE as well as in the FQHE which generates all edge excitations. This $\mathcal{W}_{1+\infty}$ was first introduced in the case $\nu = 1$ IQHE by Cappelli *et al* [13], describing the incompressibility of the quantum droplet. We have shown that the Laughlin wavefunctions for $\nu = 1/(2p+1)$ can be interpreted as highest-weight vectors of a $\mathcal{W}_{1+\infty}$ which describes the quantized algebra of the area-preserving diffeomorphisms. For this generalization, we have introduced an electron–electron interaction term which can be considered as adding flux quanta to each electron as in Jain's composite-fermion theory. Further, we calculated all the edge excitations of this quantum droplet, interpreting them as area-preserving surface deformations, and we could show that these are surface waves which are moving with the same velocity and in the same direction.

There exist many other examples of trial wavefunctions, not only for filling fraction $\nu = 1/(2p+1)$. We have applied our methods to wavefunctions for multi-layer systems and systems of interacting Landau levels.

An open question still is how the Coulomb interaction breaks this symmetry.

Acknowledgments

We would like to thank W Nahm, A Honecker, M Rösgen and M Terhoeven for useful discussions and careful reading of the manuscript. Moreover, we thank all participants of the workshop on the quantum Hall effect in Haus Windersch (Münster im Taunus, Germany). This work is supported by the Deutsche Forschungsgemeinschaft.

Appendix

In this appendix, we sketch a derivation of equation (3.10). First, one shows inductively that

$$(b_i)^n \psi_p = \sum_{\substack{1\leqslant j_k \neq j_l \leqslant N \\ j_k, j_l \neq i \\ (1\leqslant k<l\leqslant n)}} \prod_{k=1}^{n} \frac{1}{z_i - z_{j_k}} \psi_p. \tag{A.1}$$

Thus, the action of $W_n^{(s)}$ on ψ_p is given by

$$W_n^{(s)} \psi_p = \sum_{\substack{1\leqslant j_k \neq j_l \leqslant N \\ (0\leqslant k<l\leqslant s-1)}} \prod_{k=1}^{s-1} \frac{z_{j_0}^{n+s-1}}{z_{j_0} - z_{j_k}} \psi_p. \tag{A.2}$$

Note that this expression does not explicitly depend on p. Now, we can rewrite the sums in terms of determinants in the following way:

$$\sum_{\substack{1\leqslant j_k \neq j_l \leqslant N \\ (0\leqslant k<l\leqslant s)}} \prod_{k=1}^{s} \frac{z_{j_0}^r}{z_{j_0} - z_{j_k}} \tag{A.3}$$

$$= \sum_{1\leqslant j_0<j_1<\ldots<j_s\leqslant N} \sum_{\sigma\in S_{s+1}/S_s} |S_s| \frac{z_{\sigma(j_0)}^r}{z_{\sigma(j_0)} - z_{\sigma(j_k)}} \tag{A.4}$$

$$= s! \sum_{1\leqslant j_0<j_1<\ldots<j_s\leqslant N} \sum_{l=0}^{s} (-)^{l+1} z_{j_l}^r \prod_{k=1}^{l-1} (z_{j_k} - z_{j_l})^{-1} \prod_{k=l+1}^{s} (z_{j_l} - z_{j_k})^{-1} \tag{A.5}$$

$$= s! \sum_{1\leqslant j_0<j_1<\ldots<j_s\leqslant N} \frac{\sum_{l=0}^{s} (-)^{l+1} z_{j_l}^r \prod_{\substack{m<n \\ m,n\neq l}} (z_{j_m} - z_{j_n})}{\prod_{i<k} (z_{j_i} - z_{j_k})} \tag{A.6}$$

where we sum over all fixed-point free permutations of S_{s+1} and where the sign comes from the asymmetry of the factors $(z_i - z_j)$. The last expression is nothing but the expansion of a determinant divided by a Vandermonde determinant; hence, we arrive at equation (3.10).

References

[1] Klitzing K, Dorda G and Pepper M 1980 *Phys. Rev. Lett.* **45** 494
[2] Tsui D C, Stromer H L and Gossard A C 1982 *Phys. Rev. Lett.* **48** 1559
[3] Laughlin R B 1981 *Phys. Rev.* B **23** 5632–3
[4] Laughlin R B 1983 *Phys. Rev. Lett.* **50** 1395–8
[5] Prange R E and Girvin S M 1987 *The Quantum Hall Effect* (Berlin: Springer)
[6] Stone M 1992 *Quantum Hall Effect* (Singapore: World Scientific)
[7] Haldane F D M 1983 *Phys. Rev. Lett.* **51** 605–8
[8] Halperin B I 1984 *Phys. Rev. Lett.* **52** 1583–6
[9] Jain J K 1989 *Phys. Rev. Lett.* **63** 199–202
[10] Du R R, Stromer H L, Tsui D C, Pfeifer L N and West K W 1993 *Phys. Rev. Lett.* **70** 2944
[11] Willet R L, Ruel R R, Paalanen M A, West K W and Pfeiffer L N 1993 *Phys. Rev.* B **47** 7344
[12] Halperin B I, Lee P A and Read N 1993 *Phys. Rev.* B **47** 7312
[13] Cappelli A, Trugenberger C A and Zemba G R 1993 *Nucl. Phys.* B **396** 465–90; 1993 *Phys. Lett.* **306B** 100–7
[14] Pope C N, Romans L J and Shen X 1990 *Phys. Lett.* **236B** 173; 1990 *Nucl. Phys.* B **339** 191
[15] Bakas I 1989 *Phys. Lett.* **228B** 57
[16] Witten E 1992 *Nucl. Phys.* B **373** 187–213
[17] Halperin B I 1982 *Phys. Rev.* B **25** 2185–90
[18] Wen X G 1992 *Int. J. Mod. Phys.* B **6** 1711–62
[19] Block B and Wen X G 1990 *Phys. Rev.* B **42** 8133
[20] Fröhlich J and Kerler T 1991 *Nucl. Phys.* B **354** 369
[21] Fröhlich J and Zee A 1991 *Nucl. Phys.* B **364** 517
[22] Cappelli A, Dunne G V, Trugenberger C A and Zemba G R 1993 *Nucl. Phys.* B **398** 531–67
[23] Witten E 1989 *Commun. Math. Phys.* **121** 351
[24] Stone M 1991 *Int. J. Mod. Phys.* B **5** 509
[25] Niu Q, Thouless D J and Wu Y S 1985 *Phys. Rev.* B **31** 3372–7
[26] Kohmoto M 1985 *Ann. Phys., NY* **160** 343–54
[27] Avron J F and Seiler R 1985 *Phys. Rev. Lett.* **54** 259–62
[28] Bellissard J 1986 *Lecture Notes in Physics 257* (Berlin: Springer) pp 99–156
[29] Lopez A and Fradkin E 1990 *Phys. Rev.* B **44** 5246
[30] Verlinde E 1991 *Modern Quantum Field Theory* (Singapore: World Scientific) pp 450–61
Stone M 1992 *Quantum Hall Effect* (Singapore: World Scientific) pp 258–39
[31] Balachandran A P, Bimonte G, Gupta K S ans Stern A 1992 *Int. J. Mod. Phys.* A **7** 5855–76
[32] Fubini S 1991 *Mod. Phys. Lett.* A **6** 347
[33] Cristofano C, Maiella G, Musto R and Nicodemi F 1991 *Phys. Lett.* **262B** 88; 1991 *Mod. Phys. Lett* A **6** 1779; 1991 *Mod. Phys. Lett.* A **6** 2985
[34] Moore G and Read N 1991 *Nucl. Phys.* B **360** 362
[35] Wen X G and Zee A 1992 *Preprint* NST-ITP-92-10
[36] Jain J K 1989 *Phys. Rev.* B **40** 8079–82